frequency ω_a increases cannot be due to the inhomogeneous linewidth and is instead due to a distribution of enhancement factors. Utilizing the same analysis as Mendis and Anderson,[2] we find that the room-temperature rotary-saturation data presented in Ref. 1 are consistent with a distribution of enhancement factors having a maximum enhancement factor of $\eta_{max} \simeq 5000$. The uncertainty in the value of η_{max} is not more than $\pm 20\%$.

The average value of the enhancement factor (as defined in Ref. 2) is $\eta_{av} = 1600$.

Stearns[8] has recently measured the enhancement factor for nickel metal and finds $\eta = 4000 \pm 500$ independent of temperature over the temperature range from 1.3 to 77 °K. We believe our room-temperature result combined with the low-temperature results of Stearns indicates that η is independent of the temperature from 1.3 to 300 °K. In addition to the measurements on η Stearns found that T_1 as a function of temperature obeys an equation $T_1 T^{0.8} = 6.5 \pm 1.5$ msec °K in the range of temperatures from 1.3 to 77 °K. Using Stearn's formula for T_1 to extrapolate to $T = 300$ °K, one obtains a value for T_1 that is about half the value $T_1 = 0.16$ msec which we reported in Ref. 1. This seems reasonable since Stearns actually determines the value of T_1 at the center of the domain wall whereas we measure a complicated average over the various spins in the domain wall.

[1]D. L. Cowan and L. W. Anderson, Phys. Rev. 139, A424 (1965).
[2]E. F. Mendis and L. W. Anderson, Phys. Rev. B 2, 569 (1970).
[3]A. G. Redfield, Phys. Rev. 98, 1787 (1955).
[4]A. M. Portis and A. C. Gossard, Phys. Rev. Letters 3, 164 (1959); J. Appl. Phys. 31, 2055 (1960).
[5]L. H. Bruner, J. I. Budnik, and R. J. Blume, Phys. Rev. 121, 83 (1961).
[6]R. L. Streever and L. H. Bennet, Phys. Rev. 131, 2000 (1963).
[7]P. R. Locher and S. Geschwind, Phys. Rev. Letters 11, 333 (1963).
[8]M. B. Stearns, Bull. Am. Phys. Soc. 16, 403 (1971).

PHYSICAL REVIEW B VOLUME 4, NUMBER 7 1 OCTOBER 1971

Ising Model with Four-Spin Interactions*

F. W. Wu
Department of Physics, Northeastern University, Boston, Massachusetts 02115
(Received 3 May 1971)

It is shown that Baxter's recent results on a lattice-statistical model lead to the solution of an Ising model with two- and four-spin interactions. Critical properties of this Ising model in various regions of the parameter space are given. It is argued that four-spin or crossing interactions in a two-dimensional Ising model would in general lead to a critical exponent $\alpha' \neq 0$.

The recent exact solution by Baxter[1] of a lattice-statistical model[2] constitutes a breakthrough in the study of phase transitions. The most striking feature of Baxter's solution is that the nature of the phase transition is dependent on the energy parameters of the model. While it has been known for some time that the behavior of this lattice model is quite different in the isolated soluble cases of Ising, F, and potassium dihydrogen phosphate (KDP) models, it is for the first time that a phase transition is shown to exhibit a continuously variable exponent. Baxter's solution is given in the language of a ferroelectric model. To those who are accustomed to the "magnetic" language of phase transitions, the implications of his results are perhaps not very transparent. Therefore, we wish to point out in this note the conclusions on the more familiar Ising model that can be deduced from Baxter's solution.

It can be shown[3] that the ferroelectric problem considered in Refs. 1 and 2 is equivalent to an Ising model in zero magnetic field with *finite* two- and four-spin interactions.[4] The equivalent Ising lattice, shown in Fig. 1, has first-neighbor interactions $-J_1$ and $-J_2$, second-neighbor interactions $-J$ and $-J'$, and a four-spin interaction $-J_4$ between any four spins surrounding a unit square. The Hamiltonian reads, in obvious summation notations,

$$H = -J_1 \sum \sigma\sigma' - J_2 \sum \sigma\sigma' - J \sum \sigma\sigma' - J' \sum \sigma\sigma' - J_4 \sum \sigma\sigma'\sigma''\sigma''' . \quad (1)$$

The energy parameters of the ferroelectric problem turn out to be, using Baxter's notation,[5]

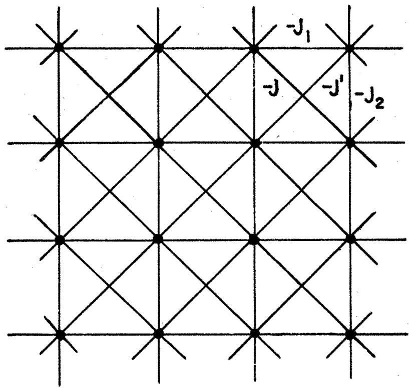

FIG. 1. Ising lattice specified by the Hamiltonian (1). Each dot denotes a spin and the four-spin interactions are not shown.

$$\begin{aligned}
\epsilon_1 &= \epsilon_2 = -J - J' - J_4 \,, \\
\epsilon_3 &= \epsilon_4 = J + J' - J_4 \,, \\
\epsilon_5 &= \epsilon_6 = J' - J + J_4 \,, \\
\epsilon_7 &= \epsilon_8 = -J' + J + J_4 \,,
\end{aligned} \tag{2a}$$

with an external electric field

$$(h, v) = (J_2, J_1). \tag{2b}$$

Baxter solved the ferroelectric model with $h = v = 0$. In the Ising language this corresponds to deleting the first-neighbor interactions J_1 and J_2. Therefore, the resulting Ising lattice is composed of two superimposed square lattices which are coupled together via four-spin interactions. If the four-spin interaction vanishes ($J_4 = 0$), the problem reduces to that of the simple square Ising lattice and can be solved by standard means.[6] The crux of the matter is that Baxter's solution can be adapted to this Ising model ($J_1 = J_2 = 0$) for arbitrary J_4! This will be the first time that a solution is found for an Ising model with many-spin interactions.

After the hard work has been done by Baxter, it is now quite straightforward to transcribe Baxter's results to the present Ising problem (with $J_1 = J_2 = 0$). The main results are summarized in the following.

(i) Since the partition function is invariant under the reversal of the signs of J_4 and J (or J_4 and J'), we may, without loss of generality, consider only $J_4 > 0$. In Fig. 2 we show the various regions in the J-J' plane defined by (a given vertex energy is favored within a region)

$$\begin{aligned}
&\epsilon_1 < (\epsilon_3, \epsilon_5, \epsilon_7), \quad \text{region I} \\
&\epsilon_3 < (\epsilon_1, \epsilon_5, \epsilon_7), \quad \text{region II} \\
&\epsilon_5 < (\epsilon_1, \epsilon_3, \epsilon_7), \quad \text{region III} \\
&\epsilon_7 < (\epsilon_1, \epsilon_3, \epsilon_5), \quad \text{region IV.}
\end{aligned} \tag{3}$$

(ii) The transition temperature T_c is given by

$$e^{2K_4} = |\cosh(K - K')/\sinh(K + K')|, \quad \text{regions I and II}$$

$$e^{2K_4} = |\sinh(K - K')/\cosh(K + K')|, \quad \text{regions III and IV} \tag{4}$$

where $K = J/kT_c$, $K' = J'/kT_c$, and $K_4 = J_4/kT_c$. We note that $T_c = 0$ on all region boundaries.

(iii) The energy is continuous at T_c.

(iv) In regions I and II the specific heat diverges at T_c with critical components

$$0 < \alpha = \alpha' = 2 - \pi/\mu_+ < 1 \,. \tag{5}$$

In regions III and IV the specific heat is continuous while the nth ($n \geq 3$) derivative of the free energy diverges as $|T - T_c|^{\pi/\mu_- - n}$ (logarithmic divergence if $\pi/\mu_- = n$), where n is the integer defined by

$$n - 1 < \pi/\mu_- \leq n. \tag{6}$$

In (5) and (6),

$$\mu_\pm = \tfrac{1}{2}\pi \pm \sin^{-1}[\tanh(2K_4)]. \tag{7}$$

(v) The case of $J_4 = 0$, the nearest-neighbor square Ising lattice, is a singular exception for which the specific heat has a logarithmic singularity.

Several remarks are now in order. First, we note that the critical behavior of the Ising model depends on the interactions J, J', and J_4. It is also tempting to infer from the above results that, in appropriate regions in the parameter space, the four-spin interaction will in general lead to higher than second-order transitions. We wish to point out, however, that it is also possible that this peculiar behavior is an artifact of setting

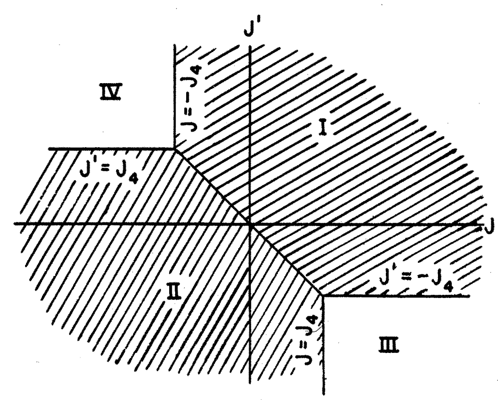

FIG. 2. Various regions in the J-J' plane for a fixed $J_4 > 0$. The phase transition is associated with an infinite specific heat in the shaded regions I and II, and is of higher than second order in regions III and IV.

$J_1 = J_2 = h = v = 0$ in the Hamiltonian (1). In the case of the F model, for example, it is known that the inclusion of a nonzero field (h, v) changes the infinite-order transition to a second-order one.[7] The inclusion of some nonzero values for J_1 and J_2 could have the same consequence in the present problem. It does appear safe, however, to infer that the inclusion of the four-spin interactions will in general *not* result in $\alpha = \alpha' = 0$.

The result that the nearest-neighbor square Ising lattice is a singular case with $\alpha = \alpha' = 0$ also appears somewhat disturbing, for it is generally believed that the critical exponents should depend only on the dimensionality of the model, and not on the range of interactions. We wish to present some counter arguments. First, some information is available at one particular point of the parameter space, namely, $J_1 = J_2 = J = J'$ and $J_4 = 0$. This is the square Ising lattice with equivalent first- and second-neighbor (crossing) interactions. For this model Domb and Dalton[8] and Dalton and Wood[9] have carried out numerical analyses on the high- and low-temperature series expansions. The study on the high-temperature series led to the critical exponent[8]

$$\gamma \cong 1.75 \ , \tag{8}$$

which does not differ from that of the nearest-neighbor planar Ising lattices. On the other hand, the study on the low-temperature series did not lead to such agreement. The authors of Ref. 9 attributed their results on the low-temperature exponents β and γ' to the erratic behavior of the Padé approximants. On reexamining their data on the first- and second-neighbor square lattice, we feel that unless something drastic happens in the high Padé approximants, it should be safe to infer the following bounds on the critical exponents β and γ':

$$0.80 < \gamma' < 1.30, \qquad 0.13 < \beta < 0.16 \ . \tag{9}$$

Accepting (9), the Rushbrook inequality $\alpha' + 2\beta + \gamma' \geq 2$ then leads to the bound

$$\alpha' \geq 0.38 \tag{10}$$

on α'. This indicates a λ transition of the type given by (5) and is definitely different from the commonly accepted value of $\alpha' = 0$ for two-dimensional lattices.[10] This result suggests that the logarithmic singularity of the nearest-neighbor Ising model is indeed a singular case. It must be noted that this is not the first time that the two-dimensional nearest-neighbor model is found to possess a unique behavior. In a recent study on the behavior of two-point correlation functions on a phase boundary, Fisher and Camp[11] showed that the planar nearest-neighbor model is unique in having a decay exponent different from the Ornstein-Zernike form. We feel that these are strong evidences which indicate that the four-spin or the crossing interactions in a planar Ising model will in general lead to a critical exponent $\alpha' \neq 0$.

---

*Work supported in part by National Science Foundation Grant No. GP-25306.

[1]R. J. Baxter, Phys. Rev. Letters 26, 832 (1971).

[2]C. Fan and F. Y. Wu, Phys. Rev. B 2, 723 (1970).

[3]The proof follows closely that given in the Appendix of F. Y. Wu, Phys. Rev. 183, 604 (1969), which will not be reproduced here. The readers are also referred to the following review article for a more comprehensive discussion: E. H. Lieb and F. Y. Wu, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, London, 1971).

[4]There exist other mappings between the ferroelectric and the Ising problems. [See, e.g., M. Suzuki and M. E. Fisher, J. Math. Phys. 12, 235 (1971); E. H. Lieb and F. Y. Wu, in Ref. 3.] These mappings would, however, lead to infinite Ising interactions in the present problem.

[5]The zero-energy level has been chosen to make $\epsilon_1 + \epsilon_3 + \epsilon_5 + \epsilon_7 = 0$.

[6]This is the case considered by F. Y. Wu, in Ref. 3.

[7]See E. H. Lieb and F. Y. Wu in Ref. 3.

[8]C. Domb and N. W. Dalton, Proc. Phys. Soc. (London) 89, 859 (1966).

[9]N. W. Dalton and D. W. Wood, J. Math. Phys. 10, 1271 (1969).

[10]We feel that the estimates on γ' in Ref. 9 are sufficient to indicate $\alpha' > 0$.

[11]M. E. Fisher and W. J. Camp, Phys. Rev. Letters 26, 565 (1971).

PHYSICAL REVIEW B VOLUME 4, NUMBER 7 1 OCTOBER 1971

ERRATUM

Enhancement of Superconductivity in Aluminum Films, J. J. Hauser [Phys. Rev. B 3, 1611 (1971)]. Figure 1 caption should read: Transition temperature of Al-Ge and Al-Al_2O_3 films as a function of low-temperature resistivity. The values Al-10 wt% (3.6-at.%) Ge and Al-10 wt% Al_2O_3 (7-at.% O) quoted in the caption correspond only to the peak in T_c as explained in the text.